Pergamon

Int. J. Solids Structures Vol. 35, No. 20, pp. 2679–2694, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)00173–X

OPTIMIZATION THEORY APPLICATION TO SLITTED PLATE BENDING PROBLEMS

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(*Received* 17 *January* 1997; *in revised form* 5 *June* 1997)

Abstract—The bending of a thin elastic plate having a slanted through slit is investigated. To describe the contact of the slit edges in the deformed state we introduce a non-penetration condition and set the bending problem as a variational one. From the numerical point of view this variational statement makes it possible to consider the slit plate bending problem as a constrained minimization problem and to apply various minimization algorithms to find the plate deformed state and the contact efforts along the slit. The computational examples obtained by using the interior point technique are given. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

We consider the bending problem for a thin Kirchhoff's plate having a through interior slanted slit. The term slitted plate means that the plate has not only the outer borders but the inner ones called the edges of the slit. In the non-deformed initial state the slit edges touch each other along a two-dimensional surface and we say that this one describes the form of the slit. While for the outer borders of the plate standard fastening conditions can be considered, for the slit edges it is natural to assume its moving without penetration. We call this type of geometrical constraints for the slit edge displacements the non-penetration condition. Instead of the term slit the term crack might be used in supposition that in the initial non-deformed state the last one has a null opening.

Dealing with the plane plate problem and giving the attention to the crack propagation in the existent crack theory several authors consider the boundary statement of this problem. In these papers it was assumed that the crack remains open (see for example Cherepanov, 1979).

For bending plates in the crack theory the equilibrium problem has been studied by Osadchuk (1985), Bui (1978), Mikhailov (1980), Mahmoud *et al.* (1992), etc. By analogy with the plane case boundary formulations of the bending problem were used. With this object boundary conditions for bending moments, shearing and stress forces at the crack edges were introduced. However either these conditions do not suppose yet the contact between the crack edges or the equivalence of these conditions to the non-penetration of the crack edges is not established.

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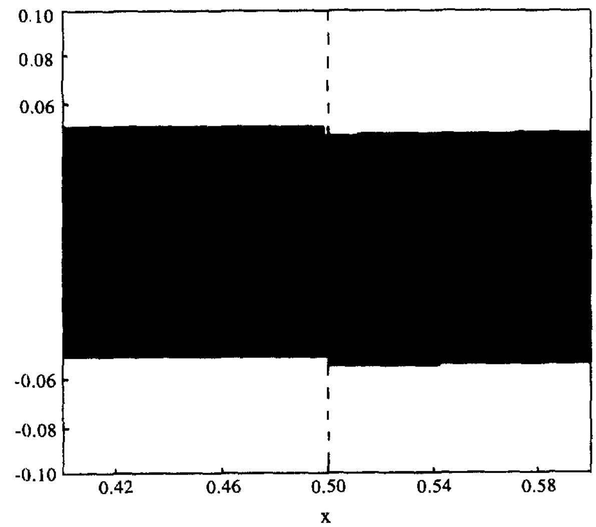

Fig. 1. Example 1 : bending of a slitted beam with non-penetration condition.

Nevertheless the penetration of the slit edges can essentially change the solution of the plate bending problem even for Kirchhoff's model. In Fig. 1 and Fig. 2 are presented the analytical solutions of slit plate bending problem in 1-D case with non-penetration condition (Kovtunenko *et al.*, 1998) and without this one. In both of these examples a unit length beam having a vertical through slit at the middle and clamped in the ends was considered. The uniformly distributed load acts in the right side of the beam. The difference of this result is obvious. As we can see, without any conditions at the slit edges the vertical displacements of the right side are big enough that the obtained solution is off the Kirchhoff's thin plate theory. Thus, for the slitted plates it seems to be more correct to formulate the bending problem taking into account the possibility of the contact of the slit edges.

In the present paper we consider the bending problem of thin plate assuming the possibility of the slit edges contact. We introduce the non-penetration condition for the case of slated slit and propose to set the slit plate bending problem as a variational one. From the numerical point of view this variational statement makes possible to consider the slit plate bending problem as a constrained minimization problem and to find the deformed state and the contact efforts along the slit by solving this problem in primal/dual variables.

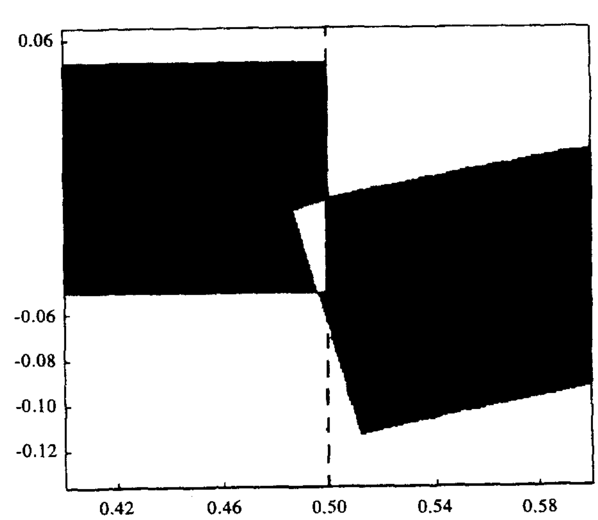

Fig. 2. Beam with a vertical slit: bending without non-penetration condition.

To do this we use Herskovits' interior point algorithm. We give some computational examples in 1-D and 2-D cases which show the fulfilment of the non-penetration condition and the suitability of the chosen numerical algorithm.

The model of the thin slit plate occurs not only in the technical applications. For example tectonic plates, described in the geophysics theory as thin elastic plates (Turkotte and Schubert, 1982), are supposed to have cracks which in general are slanted. It is assumed that tectonic plate movements in the crack locations induces the mountain formation, earthquake, etc. (Longwell *et al.*, 1969).

The non-penetration conditions for the bending plate was first proposed in Khludnev (1992) where the case of a vertical slit was considered. This condition was applied to a shell problem in Khludnev (1995a). The bending of the crack plate under an obstacle was the subject of the paper by Khludnev (1995b). In the 1-D case an analytical solution of the bending plate problem with a slit was constructed in Kovtunenko *et al.* (1998).

In the first section of our paper we propose a non-penetration condition for the slitted plate, give the variational formulation of slitted plate bending problem and derive an explicit expression for contact efforts along the slit in terms of dual variables. Section 2 is devoted to the interior point algorithm. The numerical results in 1-D and 2-D cases and the discretization techniques for the 2-D case are presented in Section 3.

2. FORMULATION OF THE SLITTED PLATE PROBLEM

Non-penetration condition

Let us denote by $W(x) = (u_1(x), u_2(x))$ and $w(x)$ horizontal and vertical displacements of the point $x = (x_1, x_2)$ of the plate middle plane Ω. Here $\Omega \subset R^2$ is a bounded domain with a smooth boundary. Let $2h$ be the thickness of the plate. According to the right normal hypothesis of Kirchhoff's plate theory (Rabotnov, 1979) the vertical displacements $w(x, z)$ of the points lying at the distance z towards the middle plane are $w(x)$; the horizontal displacements are given by the following relations:

$$W(x,z) = W(x) + z\nabla w(x), \quad |z| \leqslant h.$$

We suppose that in the non-deformed state the plate has an interior slanted slit with zero initial opening. The form of the slit is described by the regular not self-crossing surface Γ (Fig. 3). Considering the non-deformed initial state of the plate we introduce for each point $(x, z) \in \Gamma$ the normal vector $\mathbf{n}(x, z)$ and we define positive Γ^+ and negative Γ^- edges of the slit. Let us denote by $\alpha(x, z)$ the angle between $\mathbf{n}(x, z)$ and the middle plane Ω, by γ a curve formed by the intersection of Γ and Ω. We note by Ω_γ the middle plane of the slitted plate. Let $\nu(x)$ be a unit vector having the same direction as the projection of $\mathbf{n}(x, 0)$ at Ω_γ. Thus,

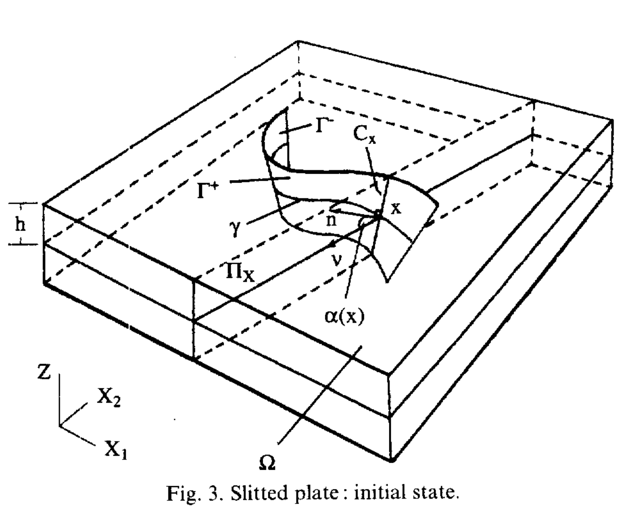

Fig. 3. Slitted plate: initial state.

$\mathbf{n}(x,0) = (v(x)\cos\alpha(x,0), \sin\alpha(x,0))$ and $v(x)$ is a normal to γ. We denote by Π_x a vertical plane passing through the point $x \in \gamma$ in the direction $\mathbf{n}(x,0)$ and by C_x an intersection of Π_x and Γ. Let us suppose that for each cross-section Π_x the curve C_x is rectilinear. Thus,

$$\mathbf{n}(x,z) = \mathbf{n}(x,0) \equiv \mathbf{n}(x), \quad \alpha(x,z) = \alpha(x,0) \equiv \alpha(x), \quad \forall z, |z| \leqslant h$$

and the coordinates $(\tilde{x}, z)$ of the surface Γ are defined by the following relations:

$$\tilde{x} = x - zv(x)\,\mathrm{tg}\,\alpha(x), \quad |z| \leqslant h, \quad x \in \gamma.$$

Let us denote by $W^+(x)$ and $w^+(x)$ [respectively by $W^-(x)$ and $w^-(x)$] the middle plane point displacements at the positive (negative) side of γ.

We assume now that if the plate middle plan displacements $w(x)$ and $W(x)$ are the order $o(h^m)$, $m \geqslant 1$ in the h-neighborhood of γ, the gradients $\nabla w(x)$, $\nabla W(x)$ are the same order with respect to h. Then, for the angle α such that $|\cos\alpha| \sim 1$, we can take the displacements at each point of positive C_x^+ (negative C_x^-) side of the C_x in the form:

$$w^{\pm}(\tilde{x}, z) = w^{\pm}(x), \quad x \in \gamma, \tag{1}$$

$$W^{\pm}(\tilde{x}, z) = W^{\pm}(x) + z\nabla w^{\pm}(x), \quad |z| < h, x \in \gamma. \tag{2}$$

This means that C_x^+ and C_x^- left to be linear after the deformation.

Indeed, under the assumption above and the Kirchhoff's hypothesis, for the part of $C_x^+ (C_x^-)$ which projection to the middle plan is in the positive (negative) side of the middle plan the difference between the exact Kirchhoff's model displacements and our approximation will be of the order $o(h^{m+1})$. Besides, for the parts of $C_x^+ (C_x^-)$, which projection is in the negative (positive) side of the middle plan, the displacements cannot be found without additional hypothesis and it seems to be appropriate the supposition that this part of the slit remains linear also.

The slit edges non-penetration condition means that for each point of Γ the projection of the difference between positive and negative edges displacements on the initial direction of the normal $\mathbf{n}(x)$ has to be non-negative. Taken $[W(x)] = W^+(x) - W^-(x)$, $[w(x)] = w^+(x) - w^-(x)$, $x \in \gamma$ and using relations (1), (2), we can write this condition in the following form:

$$([w], [W] + z[\nabla w]) \cdot \mathbf{n} \geqslant 0 \quad \forall z, |z| \leqslant h, \quad \text{for every point } x \in \gamma,$$

or

$$\Phi\left(z, W, w, \frac{\partial w}{\partial v}\right) \equiv \left[W, w, \frac{\partial w}{\partial v}\right] \cdot (\mathbf{n}, z\cos\alpha) \geqslant 0, \quad |z| \leqslant h, \quad \text{for very point } x \in \gamma. \tag{3}$$

Here $[W, w, \partial w/\partial v] \equiv ([W], [w], [\partial w/\partial v])$, $\partial w/\partial v$ is the normal derivative at γ and "$\cdot$" denotes the scalar product. Inequality (3) is linear in z; therefore if condition (3) holds for $z = \pm h$, it remains valid for all z, $|z| \leqslant h$. Thus, condition (3) can be represented in the equivalent form:

$$\Phi\left(z, W, w, \frac{\partial w}{\partial v}\right) \geqslant 0, \quad z = \pm h, \quad \text{for every point } x \in \gamma \tag{4}$$

or

$$[W]\cdot v+[w]\,\mathrm{tg}\,\alpha-h\left|\left[\frac{\partial w}{\partial v}\right]\right|\geqslant 0,\quad \text{for every point } x\in\gamma.$$

Hence taken $\alpha = 0$ the known non-penetration condition for the case of vertical slit (Khludnev, 1992) can be obtained:

$$[W]\cdot v\geqslant h\left|\left[\frac{\partial w}{\partial v}\right]\right|,\quad \text{for every point } x\in\gamma.$$

Thus, for thin plates condition (3) (or (4)) describes the displacements of the slanted slit edges assuming its contact (equality sign) or its deviation (strong inequality sign) in some *a priori* unknown locality, but simultaneously excepts its penetration.

Variational statement of the problem

Under Kirchhoff's hypothesis, the energy functional for an isotropic homogeneous plate whose middle plane occupies the domain Ω_γ is the following (Rabotnov, 1979):

$$J(W,w)=\tfrac{1}{2}\mathscr{A}(W,W)+\tfrac{1}{2}\mathscr{B}(w,w)-\langle(F,f),(W,w)\rangle_\Omega,$$

there $\langle\cdot,\cdot\rangle_\Omega$ denote the integration in the domain Ω_γ,

$$\mathscr{A}(W,\bar{W})=\mathscr{G}\int_{\Omega_\gamma}(u_{1,1}\bar{u}_{1,1}+u_{2,2}\bar{u}_{2,2}+\varkappa(u_{1,1}\bar{u}_{2,2}+u_{2,2}\bar{u}_{1,1})$$
$$+\frac{1-\varkappa}{2}(u_{1,2}+u_{2,1})(\bar{u}_{1,2}+\bar{u}_{2,1}))\,\mathrm{d}\Omega_\gamma,$$

$$\mathscr{B}(w,\bar{w})=\mathscr{D}\int_{\Omega_\gamma}(w,_{11}\bar{w},_{11}+w,_{22}\bar{w},_{22}+\varkappa w,_{11}\bar{w},_{22}+\varkappa w,_{22}\bar{w},_{11}$$
$$+2(1-\varkappa)w,_{12}\bar{w},_{12})\,\mathrm{d}\Omega_\gamma,$$

$\mathscr{G}=Eh/(1-\varkappa^2)$, $\mathscr{D}=Eh^3/3(1-\varkappa^2)$, E is Young's modulus, $\varkappa$ is Poisson's ratio, $0<\varkappa<0.5$, (F,f) is the external forces vector, $F=(f_1,f_2)$, $\bar{W}=(\bar{u}_1,\bar{u}_2)$. The subscripts after the comma signify the derivative with respect to corresponding coordinates.

We put the clamped conditions at the exterior boundary of the domain Ω_γ:

$$W=w=\frac{\partial w}{\partial v}=0.$$

Here the normal derivative $\partial w/\partial v$ corresponds with the exterior boundary of the domain Ω_γ. At the curve γ the non-penetration condition for the slit edges will be taken in the form (4).

Let $H^1_{0,\gamma}(\Omega_\gamma)$ by a subspace of Sobolev's space $H^1(\Omega_\gamma)$ contained the functions with zero at the exterior boundary of the domain Ω_γ and $H^2_{0,\gamma}(\Omega_\gamma)$ be a subspace of Sobolev's space $H^2(\Omega_\gamma)$ whose functions and its first derivatives have zero at the exterior boundary of the domain Ω_γ.

Let $H=H^1_{0,\gamma}(\Omega_\gamma)\times H^1_{0,\gamma}(\Omega_\gamma)\times H^2_{0,\gamma}(\Omega_\gamma)$ and the subset

$$\mathscr{K}=\left\{(W,w)\in H|\Phi\left(z,W,w,\frac{\partial w}{\partial v}\right)>0,\quad z\pm h\right\}.$$

where the non-penetration condition is enforced almost everywhere at γ. Let us suppose that $F\in L^2(\Omega_\gamma)\times L^2(\Omega_\gamma)$, $f\in L^2(\Omega_\gamma)$. Thus, the equilibrium problem for the bending plate having a split γ under the non-penetration condition can be formulated as a variational problem:

$$\inf_{(W,w)\in\mathscr{K}} J(W,w) = J(\overset{*}{W},\overset{*}{w}). \tag{5}$$

The functional $J(W, w)$ is coercive, weakly lower semicontinuous and strongly convex in H (see Khludnev, 1995a) and the set $\mathscr{K}$ is coincident. Thus, the minimization problem (5) has one and only one solution (Ekeland and Temam, 1979) which will be denoted by $\{\overset{*}{W},\overset{*}{w}\}$.

Contact efforts along the slit

Let us introduce the strain tensor

$$\varepsilon_{ij}(W) = \tfrac{1}{2}(u_{i,j}+u_{j,i}), \quad i,j = 1,2.$$

From Hooke's principle, the stress tensor $\sigma_{ij}(W)$, $i, j = 1, 2$ for the homogeneous isotropic plate has the form:

$$\sigma_{11}(W) = \mathscr{G}(\varepsilon_{11}(W)+\varkappa\varepsilon_{22}(W)), \quad \sigma_{22}(W) = \mathscr{G}(\varepsilon_{22}(W)+\varkappa\varepsilon_{11}(W)),$$
$$\sigma_{12}(W) = \mathscr{G}((1-\varkappa)\varepsilon_{12}(W)).$$

Let us define at γ the normal stress

$$\sigma^+(W) \equiv (\sigma_1^+(W), \sigma_2^+(w)) = (\sigma_{1j}(W^+)v_j, \sigma_{2j}(W^+)v_j),$$

the shearing stress

$$t^+(w) = \mathscr{D}\frac{\partial}{\partial v}\left(\Delta w^+ + (1-\varkappa)\frac{\partial^2 w^+}{\partial \tau^2}\right)$$

and the bending moments

$$m^+(w) = \mathscr{D}\left(\varkappa\,\Delta w^+ + (1-\varkappa)\frac{\partial^2 w^+}{\partial v^2}\right).$$

Where $\tau = (-v_2, v_1)$, $\Delta w \equiv w_{,11}+w_{,22}$. Taken W^-, w^- and $-v$ instead of v, we define by the same way the functions $\sigma^-(W)$, $t^-(w)$ and $m^-(w)$ at γ.

Let us suppose that the solution of the problem (5) is smooth enough such that

$$\sigma^\pm(W)\in L^2(\gamma)\times L^2(\gamma), \quad t^\pm(w), \quad m^\pm(w)\in L^2(\gamma).$$

Then we can consider $\Phi(z, W, w, \partial w/\partial v)$ for $z = \pm h$ as an element of the space $L^2(\gamma)$ and suppose non-penetration condition (4) fulfilled almost everywhere at γ. By analogy with Khludnev (1995a) we can show that at γ

$$[\sigma(\overset{*}{W})] = [t(\overset{*}{w})] = [m(\overset{*}{w})] = 0$$

and denote

$$\overset{*}{\sigma} \equiv \sigma^\pm(\overset{*}{W}), \quad \overset{*}{t} \equiv t^\pm(\overset{*}{w}), \quad \overset{*}{m} \equiv m^\pm(\overset{*}{w}).$$

For the case in question the existence of a solution of problem (5) is equivalent (Ekeland

and Temam, 1979) to the existence of Lagrange multipliers $\mu, \eta \in L^2(\gamma)$, $\mu \geq 0$, $\eta \geq 0$ almost everywhere at γ, such that

$$J(\overset{*}{W}, \overset{*}{w}) - \left\langle \mu, \Phi\left(h, \overset{*}{W}, \overset{*}{w}, \frac{\partial \overset{*}{w}}{\partial v}\right)\right\rangle_\gamma - \left\langle \eta, \Phi\left(-h, \overset{*}{W}, \overset{*}{w}, \frac{\partial \overset{*}{w}}{\partial v}\right)\right\rangle_\gamma$$

$$\leq J(W, w) - \left\langle \mu, \Phi\left(h, W, w, \frac{\partial w}{\partial v}\right)\right\rangle_\gamma - \left\langle \eta, \Phi\left(-h, W, w, \frac{\partial w}{\partial v}\right)\right\rangle_\gamma, \quad \forall (W, w) \in H, \quad (6)$$

moreover,

$$\left\langle \mu, \Phi\left(h, \overset{*}{W}, \overset{*}{w}, \frac{\partial \overset{*}{w}}{\partial v}\right)\right\rangle_\gamma = 0,$$

$$\left\langle \eta, \Phi\left(-h, \overset{*}{W}, \overset{*}{w}, \frac{\partial \overset{*}{w}}{\partial v}\right)\right\rangle_\gamma = 0$$

where $\langle \cdot, \cdot \rangle_\gamma$ signify the integration by the curve γ. By using Green's formula the minimum condition (6) might be formulated in the following form:

$$\langle (A\overset{*}{W}, B\overset{*}{w}) - (F, f), (W, w) \rangle_\Omega + \left\langle (\overset{*}{\sigma}, \overset{*}{t}, \overset{*}{m}), \left[W, w, \frac{\partial w}{\partial v}\right]\right\rangle_\gamma$$

$$- \left\langle \mu, \Phi\left(h, W, w, \frac{\partial w}{\partial v}\right)\right\rangle_\gamma - \left\langle \eta, \Phi\left(-h, W, w, \frac{\partial w}{\partial v}\right)\right\rangle_\gamma = 0, \quad \forall (W, w) \in H. \quad (7)$$

Here the operators A and B are the following:

$$A\overset{*}{W} = -(\sigma_{1j,j}(\overset{*}{W}), \sigma_{2j,j}(\overset{*}{W})), \quad B\overset{*}{w} = \mathscr{D}\Delta^2 \overset{*}{w}.$$

The variational equality (7) means that everywhere in Ω_γ the equilibrium equations hold:

$$-\sigma_{ij,j}(\overset{*}{W}) = f_i, \quad i,j = 1,2, \quad \mathscr{D}\Delta^2 \overset{*}{w} = f \quad (8)$$

and everywhere at the curve γ the following relation is fulfilled:

$$(\overset{*}{\sigma}, \overset{*}{t}, \overset{*}{m}) \equiv \mu(n, h\cos\alpha) + \eta(n, -h\cos\alpha). \quad (9)$$

Thus, solving the problem (5) in primal (W, w) and dual (μ, η) variables we find the deformed state of slitted plate and the contact efforts along the slit.

3. INTERIOR POINT ALGORITHM

Performing the discretization of the variational problem (5) we get the following mathematical programming problem:

Find

$$y_h \in H_h$$

such that

$$J_h(y_h) \leq J_h(\bar{y}_h), \quad \forall \bar{y}_h \in \mathscr{K}_h,$$

where

$$J_h(y_h) = \frac{1}{2} S_h y_h y_h - F_h y_h.$$

Here $y_h = (u_1, u_2, w)_h$ is the discretization of the displacement vector, H_h is the discrete analog of H, $\mathscr{K}_h$ is the discretization of $\mathscr{K}$, S_h is the stiffness matrix corresponding to our variational problem, $F_h = (F, f)_h$ is the discretization of the external force vector.

The set of constraints $\mathscr{K}_h$ might be presented as follows,

$$A_h y_h - s_h \leqslant 0,$$

where A_h is a matrix describing the constraints. Thus finally we get our problem in the form:

$$\begin{cases} \inf\limits_{y_h \in H_h} J_h(y_h), \\ A_h y_h - s_h \leqslant 0. \end{cases} \tag{10}$$

The algorithm presented below is based on a general technique for interior point algorithms in nonlinear constrained optimization, proposed by J. Herskovits (Herskovits, 1991; Herskovits, 1986), that solves Karush–Kuhn–Tucker (KKT) first order optimality conditions. To obtain these algorithms, we define Newton-like iterations to solve the nonlinear system in the primal/dual variables (y_h, λ_h) given by the equalities in KKT conditions. Then, these iterations are slightly modified in such a way to have the inequalities in these conditions satisfied at each iteration.

The algorithms require an initial estimate of y_h, at the interior of the feasible region defined by the inequality constraints, and generate a sequence of points also at the interior of this set. They are feasible directions algorithms, since at each iteration they defined a search direction that is feasible direction with respect to the inequality constraints and a descent direction of the objective, or another appropriate function. When only inequality constraints are considered, the objective is reduced at each iteration.

In what follows we omit the index $(\cdot_h)$. We call the constraints $g(y) \equiv Ay - s$, where $g \in R^m$. Then $\nabla g(y) = A$, diag $[g_i(y)] = G$, diag $[\lambda_i] = \Lambda$, $i = 1, 2, \ldots m$ and λ are the Lagrange multipliers of the minimization problem.

Applying the KKT conditions to problem (10) we have:

$$Sy - F + A^T \lambda = 0, \tag{11}$$

$$(Ay - g)^T \lambda = 0, \tag{12}$$

$$\lambda \geqslant 0, \tag{13}$$

$$Ay - g \leqslant 0. \tag{14}$$

That consists in determining the pair (y, λ), such that the eqns (11)–(14) is verified. Considering that the subscripts (0) refer to the actual values of the iteration, the iterative technique of Newton applied to these equations leads to

$$\begin{pmatrix} S & A^T \\ \Lambda A & G \end{pmatrix} \begin{pmatrix} y_0 - y \\ \lambda_0 - \lambda \end{pmatrix} = - \begin{pmatrix} \nabla L \\ G\lambda \end{pmatrix}. \tag{15}$$

Where $L(y, \lambda)$ is the Lagrangian of the problem (10). Solving the matrix system (15), for $y_0 - y = d_0$:

$$Sd_0 + A^T(\lambda_0 - \lambda) = -C - A^T\lambda, \tag{16}$$

$$\Lambda A d_0 + G(\lambda_0 - \lambda) = -G\lambda. \tag{17}$$

Solving the eqns (16) and (17):

$$\lambda_0 = -W^{-1}AS^{-1}C,$$
$$d_0 = -S^{-1}(R+C).$$

Where $W = AS^{-1}A^T - \Lambda^{-1}G$ and $R = -A^TW^{-1}AS^{-1}C$. To realize these operations, it is necessary to assure the positiveness of W and S. This result and the convergence of the proposed algorithm is proved in Auatt *et al.* (1996).

Thus, in order to simplify, the following abstract of the iterative algorithm is presented:

Step 1. Initialization. Choose: $\{y_j\}$, γ^0, ε_e, ε_d,

$$\lambda_j = \frac{1}{\tau}, \quad \tau = \begin{cases} \max\{g_j\}, & \text{if } g_j \neq 0, \quad j = 1 \ldots m, \\ 1, & \text{if } g_j = 0. \end{cases}$$

Step 2. Search Direction. Compute:

$$\begin{cases} W = AS^{-1}A^T - \Lambda^{-1}G, \\ R = -A^TW^{-1}AS^{-1}C, \end{cases} \qquad \begin{cases} \lambda_0 = -W^{-1}AS^{-1}C, \\ d = -S^{-1}(R+C). \end{cases}$$

Step 3. Convergence.

$$\begin{cases} \text{If } \|C + A^T\lambda_0\|_\infty \leqslant \varepsilon_e, \quad \text{or} \quad \text{if } \|d\|_\infty \leqslant \varepsilon_d \mapsto \text{end}, \\ \text{otherwise} \mapsto \text{continue}. \end{cases}$$

Step 4. Line Search. Compute: $\gamma_{\min} = \min\{\gamma^0, \|d\|^2\}$,

$$t_r = \begin{cases} \infty, & \text{if } a_j d^k \leqslant 0, \quad j \in I(y), \\ \min\left\{\dfrac{(\gamma_{\min} - 1)[g_j(y^k)]}{a_j d^k}\right\}, & \text{if } a_j d^k > 0, \quad j \in I(y). \end{cases}$$

$$\bar{t}_f = \frac{F^T d - y^T S d}{d^T S d},$$

make:

$$t_f = \min\{\bar{t}_f, 1\},$$
$$t = \min\{t_r, t_f\}.$$

Step 5. Updates. Make: $y_j \leftarrow y_j + td, j = 1 \ldots m$. $\lambda_j \leftarrow \max\{\lambda_{0j}, \gamma_{\min}\lambda_0^{\max}\}, j = 1 \ldots m$. Go to Step 2.

4. NUMERICAL RESULTS

1-D Case

We present here two test problems in order to compare the numerical results obtained by the interior point algorithm with the analytical solutions. In both of these examples a unit length beam of thickness $2h = 0.1$ is considered. The beam has a slit at the point $x = 0.5$ and is clamped at the endpoints $x = 0$ and $x = 1$. We take $\mathcal{G} = 1$.

The discretization of the problem was performed by the finite element method with the linear approximations for the horizontal displacements and the hermitian cube splines for the vertical displacements. The step of the discretization is $\delta = 0.01$. We apply the interior point algorithm with null initial iteration values and $\gamma^0 = 0.01$, $\varepsilon_e = \varepsilon_d = 10^{-8}$.

Example 1

The angle of the slit is $\alpha = 0$. The distributed exterior force acting at the right side of the beam has the vertical component only:

$$F(x) = 0, \quad f(x) = \begin{cases} 0, & x \in [0, 1/2), \\ C, & x \in (1/2, 1], \end{cases} \quad C = -1.$$

The analytical solution (u_a, w_a) (Kovtunenko *et al.* 1998) is

$$u_a(x) = \frac{Ch}{48(1+h^2)} \begin{cases} x, & x \in (0, 1/2), \\ x-1, & x \in (1/2, 1), \end{cases}$$

$$w_a(x) = \frac{C}{48} \begin{cases} -\dfrac{h^2}{2(1+h^2)} x^2, & x \in (0, 1/2), \\ 2(x-1)^4 + 4(x-1)^3 + \left(3 - \dfrac{h^2}{2(1+h^2)}\right)(x-1)^2, & x \in (1/2, 1). \end{cases}$$

The deformed state of the beam is presented in Fig. 1. The numerical solution (u_n, w_n) was obtained after four iterations with accuracy

$$\max \left| \frac{u_n - u_a}{u_a} \right| = 4.82 \cdot 10^{-2}, \quad \max \left| \frac{w_n - w_a}{w_a} \right| = 2.61 \cdot 10^{-2}.$$

For the contact efforts at the slit we have:

$$|\overset{*}{\sigma}| = 0.11 \cdot 10^{-2}; \quad \overset{*}{t} = 0; \quad |\overset{*}{m}| = 0.55 \cdot 10^{-4} \quad \text{(numerical results)}$$

and

$$|\overset{*}{\sigma}| = 0.10 \cdot 10^{-2}; \quad \overset{*}{t} = 0; \quad |\overset{*}{m}| = 0.52 \cdot 10^{-4} \quad \text{(analytical results)}.$$

Let us note that in the same time the solution of the bending problem without non-penetration condition (Fig. 2) gives us the slit edges free of efforts.

Example 2

The slit angle is $\alpha = \pi/8$. The distributed exterior force has the horizontal component only:

$$F(x) = \begin{cases} -C, & x \in (0, 1/2), \\ C, & x \in (1/2, 1), \end{cases} \quad C = 0.05, \quad f(x) = 0.$$

The analytical solution (u_a, w_a) (Kovtunenko *et al.* 1998) is

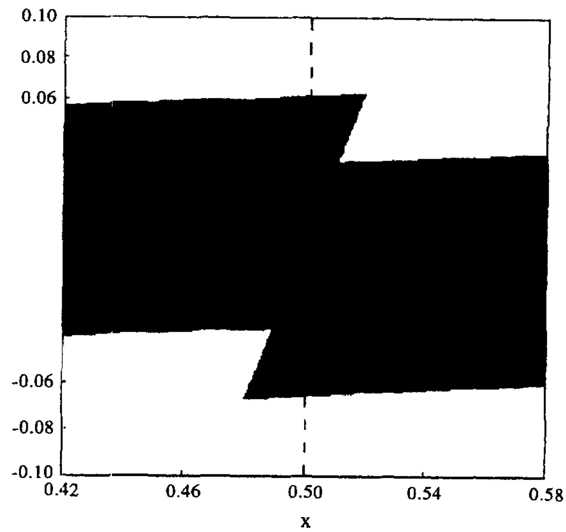

Fig. 4. Example 2: beam with a slanted slit under horizontal contractive loading.

$$u_a(x) = \frac{C}{2}\begin{cases} -x^2+\left(1-\dfrac{\theta}{6\rho}\right)x, & x\in(0,1/2), \\ x^2-\left(1+\dfrac{\theta}{6\rho}\right)x+\dfrac{\theta}{6\rho}, & x\in(1/2,1), \end{cases}$$

$$w_a(x) = \frac{C\,\mathrm{tg}\,\alpha}{4\rho}\begin{cases} 2x^3-3x^2, & x\in(0,1/2), \\ 2x^3-3x^2+1, & x\in(1/2,1). \end{cases}$$

Here $\theta = 12h^2$, $\rho = 4h^2+\mathrm{tg}^2\,\alpha$. The Fig. 4 shows the state of the beam after the deformation. For the numerical solution (u_n, w_n) we have after four iterations:

$$\max\left|\frac{u_n-u_a}{u_a}\right| = 1.44\cdot 10^{-2}, \quad \max\left|\frac{w_n-w_a}{w_a}\right| = 3.36\cdot 10^{-2}.$$

For the contact efforts at the slit we have:

$$|\overset{*}{\sigma}| = 6.0012\cdot 10^{-4}; \quad |\overset{*}{t}| = 2.4432\cdot 10^{-4}; \quad \overset{*}{m} = 0 \quad \text{(numerical results)}$$

and

$$|\overset{*}{\sigma}| = 6.8843\cdot 10^{-4}; \quad |\overset{*}{t}| = 2.8516\cdot 10^{-4}; \quad \overset{*}{m} = 0 \quad \text{(analytical results).}$$

2-D Case

A square plate of side length $L = 1.60$ m large and thickness $2h = 0.08$ m, having a linear slit $l = 0.08$ m long is considered (see Fig. 5). The exterior borders are clamped. We take $\varkappa = 0.25$ and $E = 100$ GPa and load the plate symmetrically with respect to x_2.

Discretization

We performed the discretization by 3-node triangular elements named Discrete Kirchhoff Triangle (DKT), modeling the flexural behavior (Cook *et al.*, 1989; Batoz *et al.*, 1980) and the Constant-Strain Triangle (CST), modeling the membrane behavior (Batoz *et al.*, 1980).

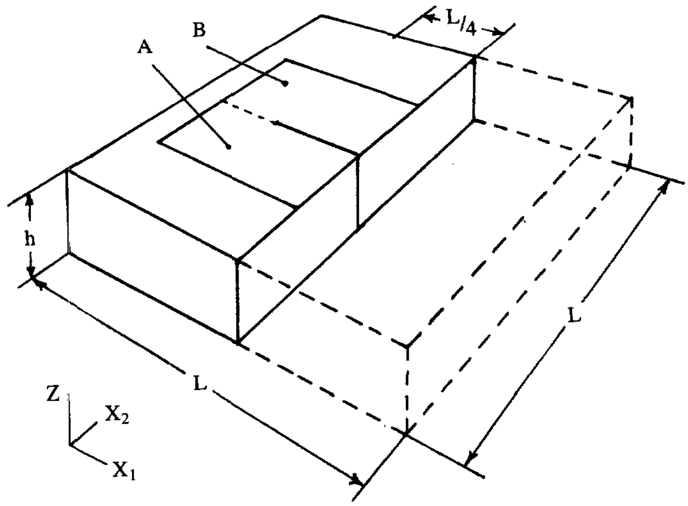

Fig. 5. Example 3, 4: plate with a linear central slit.

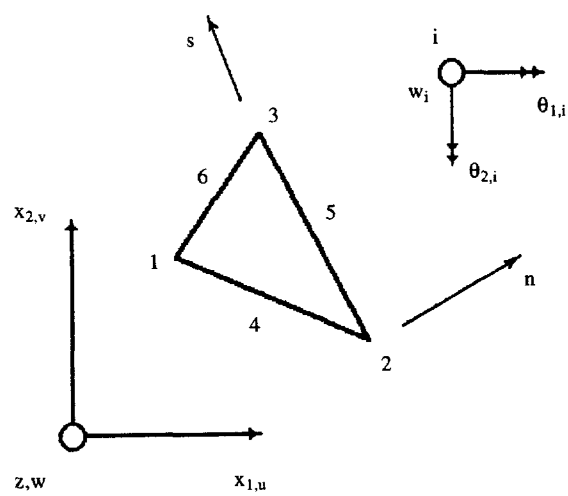

Fig. 6. Discrete Kirchhoff Triangle (DKT).

The starting point for DKT is a triangular element with corner and midside nodes, see Fig. 6. Rotations θ_1 and θ_2 are interpolated from 12 nodal rotations $\theta_{1,i}$ and $\theta_{2,i}$, $i = 1, \ldots, 6$ using a complete quadratic polynomial:

$$\theta_1 = \Sigma N_i \theta_{1,i} \quad \text{and} \quad \theta_2 = \Sigma M_i \theta_{2,i}.$$

The vertical deflection w is assumed to be cubic in an edge tangent coordinate s. The three rotation equations $w_{,si}$ at the midside nodes are written in terms of w_i, $w_{,1i}$, $w_{,2i}$ the nine degrees of freedom at corner nodes. There are a total of 21 degrees of freedom, that must be reduced to nine. The 12 degrees of freedom θ_{1i} and θ_{2i} at nodes 1–6 must be expressed in terms of w_i, $w_{,1i}$, $w_{,2i}$ at only the corner nodes (Fig. 7). With this purpose the following 12 constants are applied:

(i) transverse shear strains t_{2z} and t_{z1} vanish at corner nodes:

$$\theta_{1i} = w,_{1i} \quad \text{and} \quad \theta_{2i} = w,_{2i}, \quad i = 1, 2, 3;$$

(ii) transverse shear strain t_{sz} vanishes at midnodes:

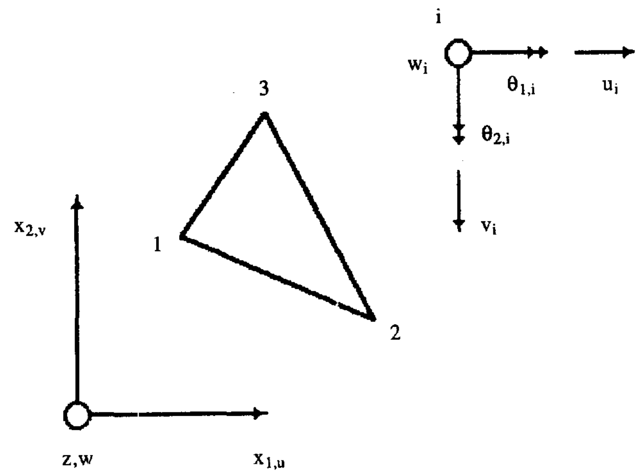

Fig. 7. Constant-Strain Triangle (CST).

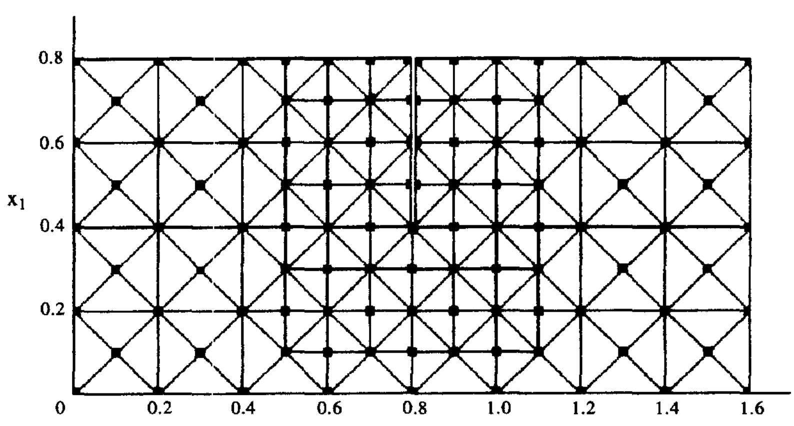

Fig. 8. Example 3, 4: triangular element discretization for a plate with linear central slit.

$$\theta_{si} = w_{,si}, \quad i = 1, 4, 6;$$

(iii) normal slopes vary linearly along each edge:

$$\theta_{nk} = \tfrac{1}{2}(w_{,ni} + w_{,nj}), \ (i, j, k) = (1, 2, 4) = (2, 3, 5) = (3, 1, 6).$$

Concerning the CST membrane behavior model, the displacements u and v are linearly interpolated by the six nodal degrees of freedom u_i and v_i, see Fig. 7.

The mesh for one-half of the plate is presented in Fig. 8. We start the interior point algorithm with null values at the initial iteration and $\gamma^0 = 0.01$, $\varepsilon_e = \varepsilon_d = 10^{-8}$. The convergence to the numerical solution was obtained in both examples below after 4–5 iterations.

Example 3

The angle of the slit is $\alpha = 0$. The external force uniformly distributed on the rectangular domain A (see Fig. 5) has only the vertical component f. The deformed state of the plate is

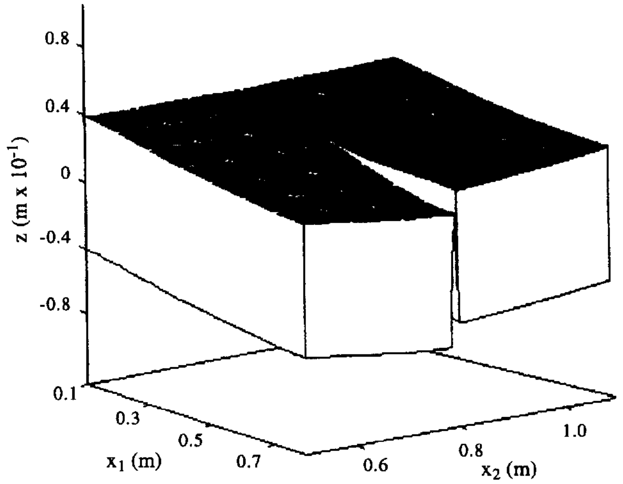

Fig. 9. Example 3 : slitted plate subjected to vertical loading.

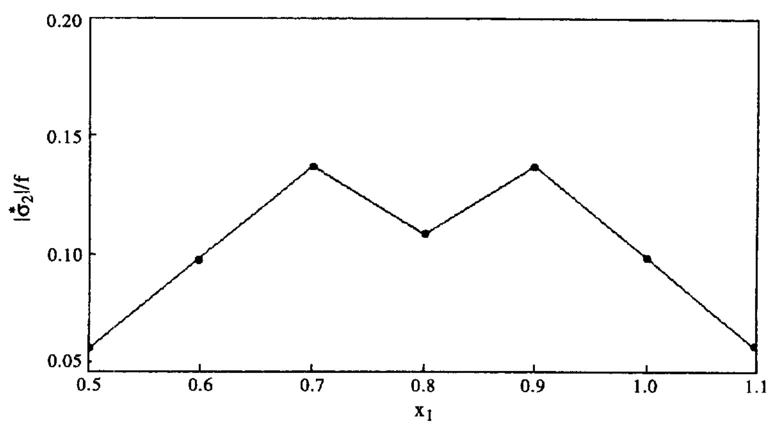

Fig. 10. Example 3 : contact efforts diagram.

presented in Fig. 9. For the contact efforts we have : $\overset{*}{\sigma}_1 = 0$; $\overset{*}{t} = 0$; $\overset{*}{m} = \overset{*}{h} \cdot \overset{*}{\sigma}_2$; the diagram of $|\overset{*}{\sigma}_2|/|f|$ is given in Fig. 10.

Example 4

The angle of the slit $\alpha = -\pi/8$. The horizontal force F is uniformly distributed on the symmetric domains A and B (see Fig. 5). The shape of the plate after deformation is presented in Fig. 11. For the contact efforts we have: $\overset{*}{\sigma}_1 = 0$; $\overset{*}{m} = 0$; the diagrams of $|\overset{*}{\sigma}_2|/|F|$ and $|\overset{*}{t}|/|F|$ are given in Fig. 12.

5. CONCLUSION

The non-penetration condition allows us to consider the slitted plate bending problem as a variational one. In turn, this variational statement makes it possible to apply the interior point algorithm to calculate the plate deformed state and the contact efforts along the slit. This algorithm requires the solution of two linear systems with the same matrix at each iteration; furthermore it takes advantage of the structure of the problem and particularities of the functions to improve calculus efficiently.

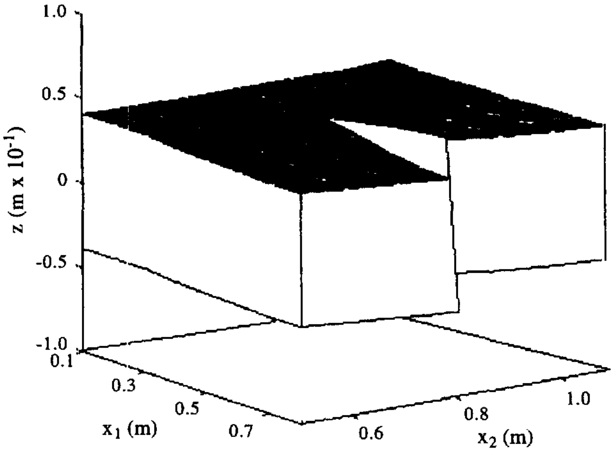

Fig. 11. Example 4: plate with a slanted slit subjected to horizontal loading.

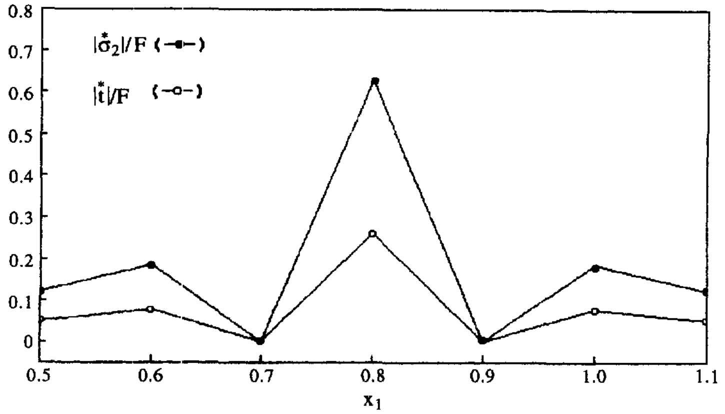

Fig. 12. Example 4: contact efforts diagram.

Acknowledgements—The first author, A.L. is grateful for the CNPq's financial support of this research during his stay as the Invited Professor at COPPE/University Federal of Rio de Janeiro, Brazil. The last author, C.E. also thanks the CNPq for her research program at PUC-Rio, Brazil, support.

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